Certificate in Quantitative Finance

GLOBAL STANDARD IN FINANCIAL ENGINEERING

Maths Primer

This is a revision course designed to act as a mathematics refresher. The volume of work covered is significantly large so the emphasis is on working through the notes and problem sheets. The four topics covered are

- Calculus
- Linear Algebra
- Differential Equations
- Probability & Statistics

1 Introduction to Calculus

1.1 Basic Terminology

We begin by defining some mathematical shorthand and number systems

$\exists$	there exists	$\longrightarrow$	which gives	$\equiv$	equivalent
$\forall$	for all	s.t	such that	$\sim$	similar
$\therefore$	therefore	:	such that	$\in$	an element of
$\because$	because	iff	if and only if	$!x$	a unique x

Natural Numbers $\mathbb{N} = \{0, 1, 2, 3, ...\}$

Integers $(\pm\mathbb{N})$ $\mathbb{Z} = \{0,\ \pm 1,\ \pm 2,\ \pm 3,\ .....\}$

Rationals $\frac{p}{q} : p, q \in \mathbb{Z}$; $\mathbb{Q} = \left\{\frac{1}{2},\ 0.76,\ 2.25,\ 0.3333333....\right\}$

Irrationals $\overline{\mathbb{Q}} = \left\{\sqrt{2},\ 0.01001000100001...,\ \pi,\ e\right\}$

Reals $\mathbb{R}$ all the above

Complex $\mathbb{C} = \left\{x + iy : i = \sqrt{-1}\right\}$

$(a, b) = a < x < b$ *open interval*

$[a, b] = a \leq x \leq b$ *closed interval*

$(a, b] = a < x \leq b$ semi-*open/closed interval*

$[a, b) = a \leq x < b$ semi-*open/closed interval*

So typically we would write $x \in (a, b)$.

Examples

$$\begin{aligned} -\infty &< x < \infty \equiv (-\infty, \infty) \\ -\infty &< x \leq b \equiv (-\infty, b] \\ a &\leq x < \infty \equiv [a, \infty) \end{aligned}$$

1.2 Functions

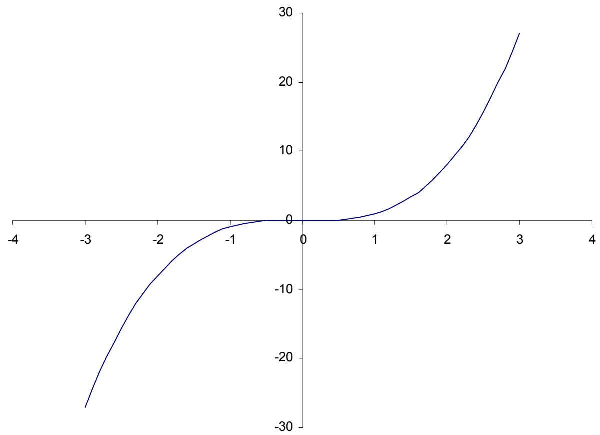
This is a term we use very loosely, but what is a function? Clearly it is a type of black box with some input and a corresponding output. As long as the correct result comes out we usually are not too concerned with what happens 'inside'.

A *function* denoted $f(x)$ of a single variable x is a rule that assigns each element of a set X (written $x \in X$) to exactly one element y of a set Y $(y \in Y)$.

A function is denoted by the form $y = f(x)$ or $x \mapsto f(x)$.

We can also write $f : X \longrightarrow Y$, which is saying that f is a mapping such that all members of the input set X are mapped to elements of the output set Y. So clearly there are a number of ways to describe the workings of a function.

For example, if $f(x) = x^3$, then $f(-2) = -2^3 = -8$.



We often write $y = f(x)$ where y is the *dependent variable* *and* x is the *independent variable.*

The set X is called the *domain of* f and the set Y is called the *image* (or *range*), written $\mathbf{Dom}f$ and **Im** f, in turn. For a given value of x there should be at most **one** value of y. So the role of a function is to operate on the domain and map it across uniquely to the range.

So we have seen two notations for the same operation.

The first $y = f(x)$ suggests a graphical representation whilst the second $f : X \longrightarrow Y$ establishes the idea of a mapping.

There are three types of mapping:

1. For each $x \in X$, $\exists$ one $y \in Y$. This is a one to one mapping (or $1-1$ function) e.g. $y = 3x + 1$.

2. More than one $x \in X$, gets mapped onto one $y \in Y$. This is a many to one mapping (or many -1 function) e.g. $y = 2x^2 + 1$, because $x = \pm 2$ yields one y.

3. For each $x \in X$, $\exists$ more than one $y \in Y$, e.g. $y = \pm\sqrt{x}$. This is a many to one mapping. Clearly it is multivalued, and has two branches. We will assume that only the positive value is being considered for consistency with the definition of a function. A one to many mapping is <u>not</u> a function.

The function maps the domain across to the range. What about a process which does the reverse? Such an operation is due to the *inverse function* which maps the image of the original function to the domain. The function $y = f(x)$ has inverse $x = f^{-1}(y)$. Interchange of x and y leads to consideration of $y = f^{-1}(x)$.

The inverse function $f^{-1}(x)$ is defined so that

$$f\left(f^{-1}(x)\right) = x \text{ and } f^{-1}(f(x)) = x.$$

Thus x^2 and $\sqrt{x}$ are inverse functions and we say they are *mutually inverse*. Note the inverse $\sqrt{x}$ is multivalued unless we define it such that only non-negative values are considered.

Example 1: What is the inverse of $y = 2x^2 - 1$.

i.e. we want y^{-1}. One way this can be done is to write the function above as

$$x = 2y^2 - 1$$

and now rearrange to have $y = ...$ so

$$y = \sqrt{\frac{x+1}{2}}.$$

Hence $y^{-1}(x) = \sqrt{\frac{x+1}{2}}$. Check:

$$yy^{-1}(x) = 2\left(\sqrt{\frac{x+1}{2}}\right)^2 - 1 = x = y^{-1}y(x)$$

Example 2: Consider $f(x) = 1/x$, therefore $f^{-1}(x) = 1/x$

$$\mathbf{Dom} f = (-\infty, 0) \cup (0, \infty) \text{ or } \mathbb{R} - \{0\}$$

Returning to the earlier example

$$y = 2x^2 - 1$$

clearly $\mathbf{Dom} f = \mathbb{R}$ (clearly) and for

$$y^{-1}(x) = \sqrt{\frac{x+1}{2}}$$

to exist we require the term inside the square root sign to be non-negative, i.e. $\frac{x+1}{2} \geq 0 \implies x > -1$, therefore $\mathbf{Dom} f = \{[-1, \infty)\}$.

An *even function* is one which has the property

$$f(-x) = f(x)$$

e.g. $f(x) = x^2$.

$f(x) = x^3$ is an example of an *odd function* because

$$f(-x) = -f(x).$$

Most functions are neither even nor odd but every function can be expressed as the sum of an even and odd function.

1.2.1 Explicit/Implicit Representation

When we express a function as $y = f(x)$, then we can obtain y corresponding to a (known) value of x. We say y is an *explicit* function. All known terms are on the right hand side (rhs) and unknown on the left hand side (lhs). For example

$$y = 2x^2 + 4x - 16 = 0$$

Occasionally we may write a function in an *implicit* form $f(x, y) = 0$, although in general there is no guarantee that for each x there is a unique y. A trivial example is $y - x^2 = 0$,which in its current form is implicit. Simple rearranging gives $y = x^2$ which is explicit.

A more complex example is $4y^4 - 2y^2x^2 - yx^2 + x^2 + 3 = 0$.

This can neither be expressed as $y = f(x)$ or $x = g(y)$.

So we see all known and unknown variables are bundled together. An implicit form which does not give rise to a function is

$$y^2 + x^2 - 16 = 0.$$

This can be written as

$$y = \sqrt{16 - x^2}.$$

and e.g. for $x = 0$ we can have either $y = 4$ or $y = -4$, i.e. one to many.

1.2.2 Types of function $f(x)$

Polynomials are functions which involve powers of x,

$$\begin{aligned} y &= f(x) = a_0 + a_1 x + a_2 x^2 + \ldots.. \\ &\quad .. + a_{n-1} x^{n-1} + a_n x^n. \end{aligned}$$

The highest power is called the *degree* of the polynomial - so $f(x)$ is an n^{th} degree polynomial. We can express this more compactly as

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

where the coefficients of x are constants.

Polynomial equations are written $f(x) = 0$, so an n^{th} degree polynomial equation is

$$a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_2 x^2 + a_1 x + a_0 = 0.$$

$k = 1, 2$ gives a linear and quadratic in turn. The most general form of quadratic equation is

$$ax^2 + bx + c = 0.$$

To solve we can complete the square which gives

$$\begin{aligned}
\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a^2} + \frac{c}{a} &= 0 \\
\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2}{4a^2} - \frac{c}{a} = \tfrac{b^2 - 4ac}{4a^2} \\
x + \tfrac{b}{2a} &= \tfrac{\pm\sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

and finally we get the well known formula for x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider:

(1) $b^2 - 4ac > 0 \longrightarrow x_1 \neq x_2 \in \mathbb{R}$: 2 distinct real roots

(2) $b^2 - 4ac = 0 \longrightarrow x = x_1 = x_2 = -\dfrac{b}{2a} \in \mathbb{R}$: one two fold root

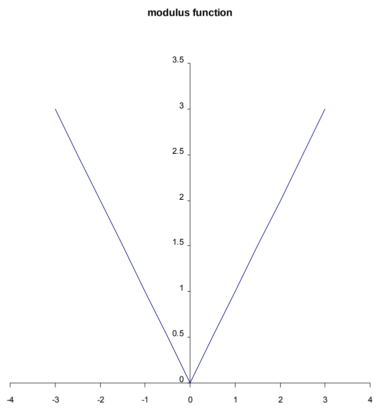
(3) $b^2 - 4ac < 0 \longrightarrow x_1 \neq x_2 \in \mathbb{C}$ − Complex conjugate pair

1.2.3 The Modulus Function

Sometimes we wish to obtain the absolute value of a number, i.e. positive part. For example the absolute value of -3.9 is 3.9. In maths there is a function which gives us the absolute value of a variable x called the *modulus function*, written $|x|$ and defined as

$$y = |x| = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases},$$

although most definitions included equality in the positive quadrant.



This is an example of a *piecewise function*.

The name is given because they are functions that comprise of 'pieces', each piece of the function definition depends on the value of x.

So, for the modulus, the first definition is used when x is non-negative and the second if x is negative.

1.3 Limits

Choose a point x_0 and function $f(x)$. Suppose we are interested in this function near the point $x = x_0$. The function need not be defined at $x = x_0$. We write $f(x) \longrightarrow l$ as $x \longrightarrow x_0$, "if $f(x)$ gets closer and closer to l as x gets close to x_0". Mathematically we write this as

$$\lim_{x \longrightarrow x_0} f(x) \longrightarrow l,$$

if $\exists$ a number l such that

- Whenever x is close to x_0
- $f(x)$ is close to l.

The limit only exists if

$$\begin{aligned} f(x) &\longrightarrow l \text{ as } x \to x_0^- \\ f(x) &\longrightarrow l \text{ as } x \to x_0^+ \end{aligned}$$

Let us have a look at a few basic examples and corresponding "tricks" to evaluate them

Example 1:

$$\lim_{x \longrightarrow 0} \left(x^2 + 2x + 3\right) \longrightarrow 0 + 0 + 3 \longrightarrow 3;$$

Example 2:

$$\lim_{x \longrightarrow \infty} \frac{x^2 + 2x + 2}{3x^2 + 4} \quad = \quad \lim_{x \longrightarrow \infty} \frac{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{2}{x^2}}{\frac{3x^2}{x^2} + \frac{4}{x^2}} =$$

$$\lim_{x \longrightarrow \infty} \frac{1 + \frac{2}{x} + \frac{2}{x^2}}{3 + \frac{4}{x^2}} \quad \longrightarrow \quad \frac{1}{3}.$$

Example 3:

$$\lim_{x \longrightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \longrightarrow 3} \frac{(x+3)(x-3)}{(x-3)} = \lim_{x \longrightarrow 3} (x+3) \longrightarrow 6$$

A function $f(x)$ is **continuous** at x_0 if

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0).$$

That is, 'we can draw its graph without taking the pen off the paper'.

1.3.1 The exponential and log functions

The *logarithm* (or simply log) was introduced to solve equations of the form

$$a^p = N$$

and we say p is log of N to base a. That is we take logs of both sides ($\log_a$)

$$\log_a a^p = \log_a N$$

which gives

$$p = \log_a N.$$

By definition $\log_a a = 1$ (important).

We will often need the exponential function e^x and the (natural) logarithm $\log_e x$ or $(\ln x)$.

Here

$$e = 2.718281828\ldots.$$

which is the approximation to

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

when n is very large. Similarly the exponential function can be approximated from

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$\ln x$ and e^x are mutual inverses:

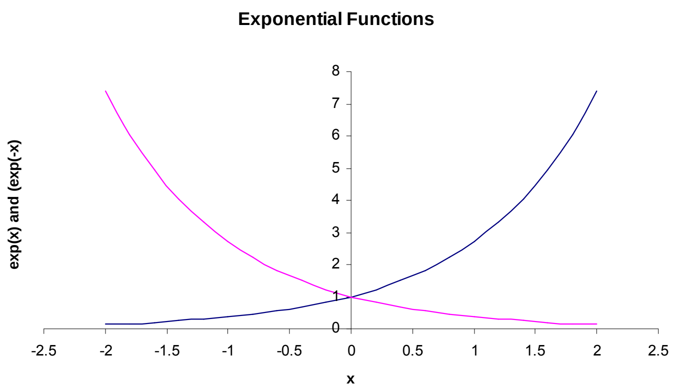
$$\log\left(e^x\right) = e^{\log x} = x.$$

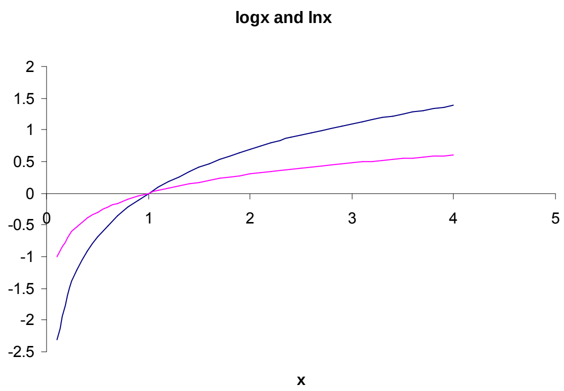
Also

$$\frac{1}{e^x} = e^{-x}.$$

Here we have used the property $(x^a)^b = x^{ab}$, which allowed us to write $\frac{1}{e^x} = (e^x)^{-1} = e^{-x}$.

Their graphs look like this:





Note that e^x is always strictly positive. It tends to zero as x becomes very large and negative, and to infinity as x becomes large and positive. To get an idea of how quickly e^x grows, note the approximation $e^5 \approx 150$.

Later we will also see $e^{-x^2/2}$, which is particularly useful in probability. This function decays particularly rapidly as $|x|$ increases.

Note:

$$e^x e^y = e^{x+y}, \qquad e^0 = 1$$

(recall $x^a.x^b = x^{a+b}$) and

$$\log(xy) = \log x + \log y, \quad \log(1/x) = -\log x, \quad \log 1 = 0.$$

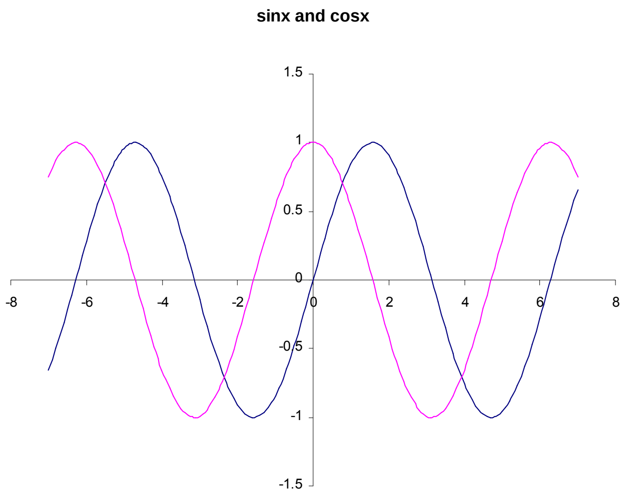
$$\log\left(\frac{x}{y}\right) = \log x - \log y.$$

$$\begin{array}{ll} \mathbf{Dom}\left(e^{x}\right) = \mathbb{R}, & \mathrm{Im}\left(e^{x}\right) = (0, \infty) \\ \mathbf{Dom}\left(\ln x\right) = (0, \infty)\,, & \mathrm{Im}\left(\ln x\right) = \mathbb{R} \end{array}$$

Example:

$$\lim_{x \to \infty} e^{-x} \longrightarrow 0; \;\; \lim_{x \to \infty} e^{x} \longrightarrow \infty; \;\; \lim_{x \to 0} e^{x} \longrightarrow e^{0} = 1.$$

1.3.2 Trigonometric/Circular Functions



$\sin x$ is an *odd* function, i.e. $\sin(-x) = -\sin x$.

It is *periodic* with period 2π: $\sin(x + 2\pi) = \sin x$. This means that after every 360° it repeats itself.

$\sin x = 0 \iff x = n\pi \ \forall n \in \mathbb{Z}$

$\mathbf{Dom}\,(\sin x) = \mathbb{R}$ and $\mathbf{Im}\,(\sin x) = [-1, 1]$

$\cos x$ is an *even* function, i.e. $\cos(-x) = \cos x$.

It is *periodic* with period 2π: $\cos(x + 2\pi) = \cos x$.

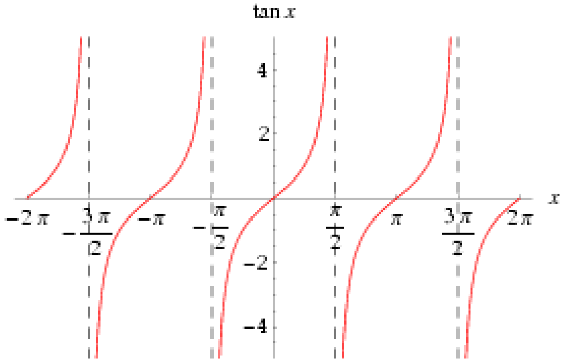
$\cos x = 0 \iff x = (2n+1)\frac{\pi}{2}\ \forall n \in \mathbb{Z}$

$\mathbf{Dom}\,(\cos x) = \mathbb{R}$ and $\mathbf{Im}\,(\cos \mathbf{x}) = [-1, 1]$

$$\boxed{\tan x = \frac{\sin x}{\cos x}}$$

This is an odd function: $\tan(-x) = \tan x$

$\boxed{\text{Periodic: } \tan(x + \pi) = \tan x}$



$$\mathbf{Dom} = \{x : \cos x \neq 0\} = \left\{x : x \neq (2n+1)\frac{\pi}{2};\ n \in \mathbb{Z}\right\} = \mathbb{R} - \left\{(2n+1)\frac{\pi}{2};\ n \in \mathbb{Z}\right\}$$

Trigonometric Identities:

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1; \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y; \ \tan(x+y) = \frac{\tan x + \tan y}{1 \mp \tan x \tan y} \end{aligned}$$

Exercise: Verify the following $\sin\left(x + \frac{\pi}{2}\right) = \cos x;\ \cos\left(\frac{\pi}{2} - x\right) = \sin x.$

The reciprocal trigonometric functions are defined by

$$\sec x = \frac{1}{\cos x}; \quad \csc x = \frac{1}{\sin x}; \quad \cot x = \frac{1}{\tan x}$$

More examples on limiting:

$$\lim_{x \longrightarrow 0} \sin x \longrightarrow 0; \quad \lim_{x \longrightarrow 0} \frac{\sin x}{x} \longrightarrow 1; \quad \lim_{x \longrightarrow 0} |x| \longrightarrow 0$$

What about $\lim_{x \longrightarrow 0} \frac{|x|}{x}$?

$$\begin{aligned} \lim_{x \longrightarrow 0^+} \frac{|x|}{x} &= 1 \\ \lim_{x \longrightarrow 0^-} \frac{|x|}{x} &= -1 \end{aligned}$$

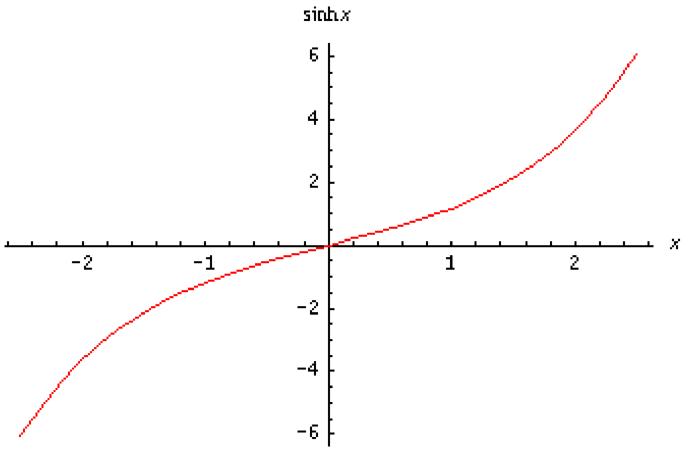
therefore $\frac{|x|}{x}$ does not tend to a limit as $x \to 0$.

Hyperbolic Functions

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

Odd function: $\sinh(-x) = -\sinh x$

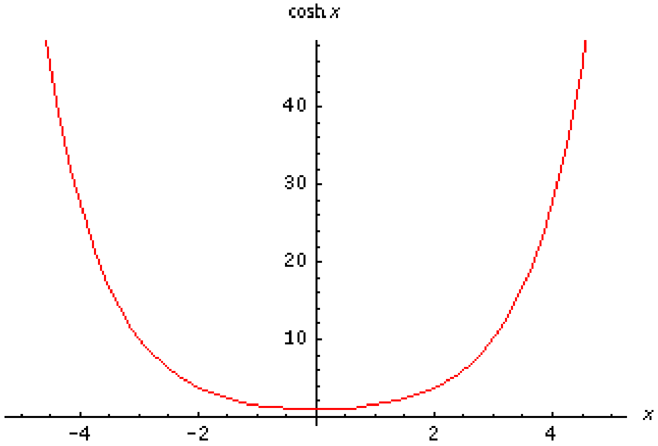
$\mathbf{Dom}(\sinh x) = \mathbb{R};\quad \operatorname{Im}(\sinh x) = \mathbb{R}$



$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

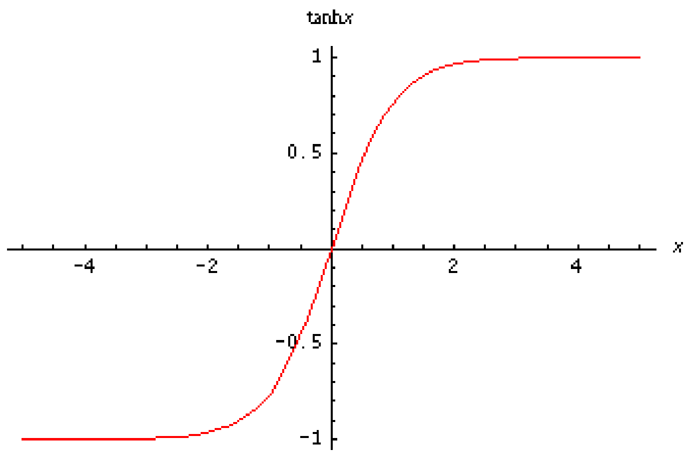
Even function: $\cosh(-x) = \cosh x$

$\mathbf{Dom}(\cosh x) = \mathbb{R}, \quad \mathrm{Im}(\cosh \mathbf{x}) = [1, \infty)$



$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\mathbf{Dom}\,(\tanh x) = \mathbb{R}; \quad \mathrm{Im}\,(\tanh x) = (-1, 1)$$



Identities:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1 \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \cosh x \cosh y + \sinh x \sinh y \end{aligned}$$

Inverse Hyperbolic Functions

$y = \sinh^{-1} x \longrightarrow x = \sinh y = \frac{\exp y - \exp(-y)}{2}$;

$2x = \exp y - \exp(-y)$

multiply both sides by $\exp y$ to obtain $2xe^y = e^{2y} - 1$ which can be written as

$$(e^y)^2 - 2x\,(e^y) - 1 = 0.$$

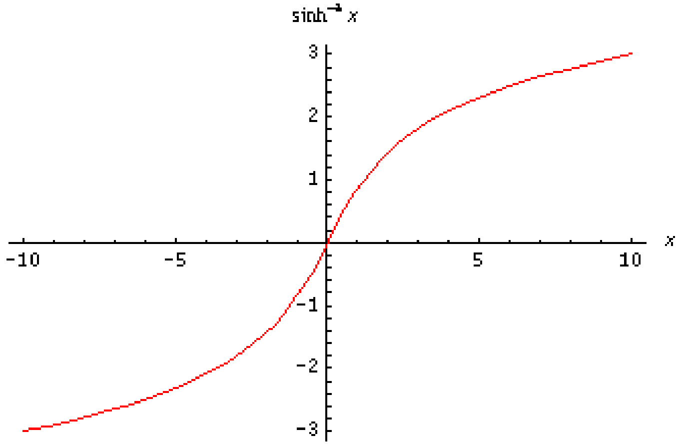
This gives us a quadratic in e^y therefore

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Now $\sqrt{x^2+1} > x \Longrightarrow x - \sqrt{x^2+1} < 0$ and we know that $e^y > 0$ therefore we have $e^y = x + \sqrt{x^2+1}$. Hence taking logs of both sides gives us

$$\sinh^{-1} x = \ln\left|x + \sqrt{x^2+1}\right|$$

$\mathbf{Dom}\left(\sinh^{-1}\mathbf{x}\right)=\mathbb{R};\quad \mathsf{Im}\left(\sinh^{-1}x\right)=\mathbb{R}$



Similarly $y=\cosh^{-1}x\longrightarrow x=\cosh y=\frac{\exp y+\exp(-y)}{2}$;

$2x=\exp y+\exp\left(-y\right)$ and again multiply both sides by $\exp y$ to obtain

$$\left(e^{y}\right)^{2}-2x\left(e^{y}\right)+1=0.$$

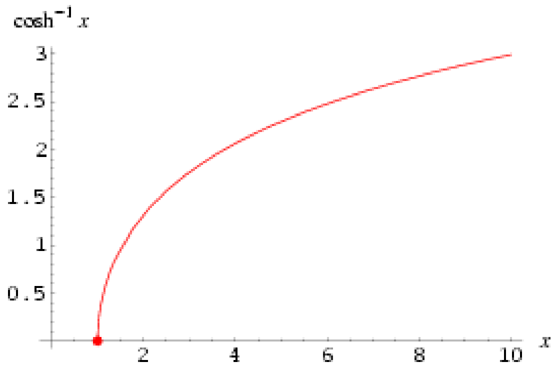
and

$$e^{y}=x+\sqrt{x^{2}-1}$$

We take the positive root (not both) to ensure this is a function.

$$\cosh^{-1} x = \ln\left|x + \sqrt{x^2 - 1}\right|$$

$$\mathbf{Dom}\left(\cosh^{-1} x\right) = [1, \infty); \quad \mathrm{Im}\left(\cosh^{-1} x\right) = [0, \infty)$$



We finish off by obtaining an expression for $\tanh^{-1} x$. Put $y = \tanh^{-1} x \longrightarrow$

$$x = \tanh y = \frac{\exp y - \exp(-y)}{\exp y + \exp(-y)};$$

$$x \exp y + x \exp(-y) = \exp y - \exp(-y)$$

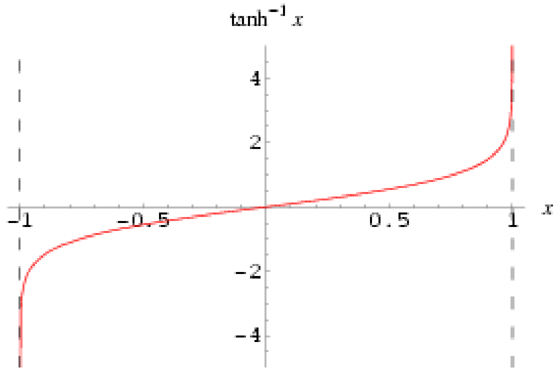
and as before multiply through by e^y

$$\begin{aligned} x\exp 2y + x &= \exp 2y - 1 \\ \exp 2y\,(1-x) &= 1 + x \longrightarrow \exp 2y = \frac{1+x}{1-x} \end{aligned}$$

taking logs gives

$$2y = \ln\left|\frac{1+x}{1-x}\right| \Longrightarrow \tanh^{-1} x = \tfrac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

$$\mathbf{Dom}\left(\tanh^{-1} x\right) = (-1,1)\,; \quad \operatorname{Im}\left(\tanh^{-1} x\right) = \mathbb{R}$$



1.4 Differentiation

A basic question asked is how fast does a function $f(x)$ change with x? The **derivative** of $f(x)$, written

$$\frac{df}{dx} : \text{ Leibniz notation}$$

or

$$f'(x) : \text{ Lagrange notation,}$$

is defined for each x as

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

assuming the limit exists (it may not) and is unique.

The term on the right hand side $\frac{f(x+\delta x)-f(x)}{\delta x}$ is called *Newton quotient*.

Differentiability implies continuity but converse does not always hold.

There is another notation for a derivative due to Newton, if a function varies with time, i.e. $y = y\,(t)$ then a dot is used

$$\dot{y}$$

We can also define operator notation due to Euler. Write

$$D \equiv \frac{d}{dx}.$$

Then D operates on a function to produce its derivative, i.e. $Df \equiv \frac{df}{dx}$.

The earlier form of the derivative given is also called a *forward derivative*. Other possible definitions of the derivative are

$$
\begin{aligned}
f'(x) &= \lim_{\delta x \to 0} \frac{1}{\delta x} \left(f(x) - f(x - \delta x) \right) \quad \textit{backward} \\
f'(x) &= \lim_{\delta x \to 0} \frac{1}{2\delta x} \left(f(x + \delta x) - f(x - \delta x) \right) \quad \textit{centred}
\end{aligned}
$$

Example: Differentiating x^3 from first principles:

$$
\begin{aligned}
f(x) &= x^3 \\
f(x + \delta x) &= (x + \delta x)^3 = x^3 + \delta x^3 + 3x\delta x (x + \delta x) \\
\frac{f(x + \delta x) - f(x)}{\delta x} &= \frac{\delta x^3 + 3x\delta x (x + \delta x)}{\delta x} = \delta x^2 + 3x^2 + 3x\delta x \\
&\longrightarrow 3x^2 \text{ as } \delta x \to 0;
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}x^n &= nx^{n-1};\ \frac{d}{dx}e^x = e^x;\ \ \frac{d}{dx}e^{ax} = ae^{ax};\\
\frac{d}{dx}\log x &= \frac{1}{x};\ \frac{d}{dx}\cos x = -\sin x;\ \frac{d}{dx}\sin x = \cos x,\ \frac{d}{dx}\tan x = \sec^2 x
\end{aligned}
$$

and so on. Take these as defined (standard results).

Examples:

$$
\begin{aligned}
f(x) &= x^5 \rightarrow f'(x) = 5x^4\\
g(x) &= e^{3x} \rightarrow g'(x) = 3e^{3x} = 3g(x)
\end{aligned}
$$

Linearity: If λ and μ are constants and $y = \lambda f(x) + \mu g(x)$ then

$$\frac{dy}{dx} = \frac{d}{dx}(\lambda f(x) + \mu g(x)) = \lambda f'(x) + \mu g'(x).$$

Thus if $y = 3x^2 - 6e^{-2x}$ then

$$dy/dx = 6x + 12e^{-2x}.$$

1.4.1 Product Rule

If $y = f(x)\, g(x)$ then

$$\frac{dy}{dx} = f'(x)\, g(x) + f(x)\, g'(x)\,.$$

Thus if $y = x^3 e^{3x}$ then

$$dy/dx = 3x^2 e^{3x} + x^3 \left(3e^{3x}\right) = 3x^2 (1+x)\, e^{3x}.$$

1.4.2 Function of a Function Rule

Differentiation is often a matter of breaking a complicated problem up into simpler components. The function of a function rule is one of the main ways of doing this.

If $y = f(g(x))$ then

$$\frac{dy}{dx} = f'(g(x))\,g'(x).$$

Thus if $y = e^{4x^2}$ then

$$dy/dx = e^{4x^2} 4.2x = 8xe^{4x^2}.$$

So differentiate the whole function, then multiply by the derivative of the "inside" $(g(x))$.

Another way to think of this is in terms of the **chain rule.**

Write $y = f(g(x))$ as

$$y = f(u), \; u = g(x).$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} f(u) = \frac{du}{dx} \frac{d}{du} f(u) = g'(x) f'(u) \\ &= g'(x) f'(g(x)). \end{aligned}$$

Symbolically, we write this as

$$\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du}$$

provided u is a function of x alone.

Thus for $y = e^{4x^2}$, write $u = 4x^2$, $y = e^u$. Then

$$\frac{dy}{dx} = \frac{du}{dx}\frac{dy}{du} = 8xe^{4x^2}.$$

Further examples:

$y = \sin x^3$

$$\begin{aligned} y &= \sin u, \text{ where } u = x^3 \\ y' &= \cos u.3x^2 \longrightarrow y' = 3x^2 \cos x^3 \end{aligned}$$

$y = \tan^2 x$: this is how we write $(\tan x)^2$ so put

$$\begin{aligned} y &= u^2 \text{ where } u = \tan x \\ y' &= 2u.\sec^2 x \longrightarrow y' = 2\tan x \sec^2 x \end{aligned}$$

$y = \ln \sin x$. Put $u = \sin x \longrightarrow y = \ln u$

$$\frac{dy}{du} = \frac{1}{u}, \quad \frac{du}{dx} = \cos x$$

hence $y' = \cot x$.

Exercise: Differentiate $y = \log \tan^2 x$ to show $\dfrac{dy}{dx} = 2 \sec x \csc x$

1.4.3 Quotient Rule

If $y = \dfrac{f(x)}{g(x)}$ then

$$\frac{dy}{dx} = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{(g(x))^2}.$$

Thus if $y = e^{3x}/x^2$,

$$\frac{dy}{dx} = \frac{x^2 3e^{3x} - 2xe^{3x}}{x^4} = \frac{3x-2}{x^3} e^{3x}.$$

This is a combination of the product rule and the function of a function (or chain) rule. It is very simple to derive:

Starting with $y = \dfrac{f(x)}{g(x)}$ and writing as $y = f(x)(g(x))^{-1}$ we apply the product rule

$$\frac{dy}{dx} = \frac{df}{dx}(g(x))^{-1} + f(x)\frac{d}{dx}(g(x))^{-1}$$

Now use the chain rule on $(g(x))^{-1}$; i.e. write $u = g(x)$ so

$$\begin{aligned}\frac{d}{dx}(g(x))^{-1} &= \frac{du}{dx}\frac{d}{du}u^{-1} = g'(x)\left(-u^{-2}\right) \\ &= -\frac{g'(x)}{g(x)^2}.\end{aligned}$$

Then

$$\frac{dy}{dx} = \frac{1}{g(x)}\frac{df}{dx} - f(x)\frac{g'(x)}{g(x)^2} = \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{g(x)^2}.$$

To simplify we note that the common denominator is $g(x)^2$ hence

$$\frac{dy}{dx} = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{g(x)^2}.$$

Examples:

$$\begin{aligned} \frac{d}{dx}(xe^x) &= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) \\ &= xe^x + e^x = e^x(x+1); \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(e^x/x) &= \frac{x(e^x)' - e^x(x)'}{(x)^2} = \frac{xe^x - e^x}{x^2} \\ &= \frac{e^x}{x^2}(x-1); \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\left(e^{-x^2}\right) &= \frac{d}{dx}(e^u) \quad \text{where} \quad u = -x^2 \therefore du = -2x dx \\ &= (-2x)\, e^{-x^2}. \end{aligned}$$

1.4.4 Implicit Differentiation

Consider the function

$$y = a^x$$

where a is a constant. If we take natural log of both sides

$$\ln y = x \ln a$$

and now differentiate both sides by applying the chain rule to the left hand side

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y \ln a \end{aligned}$$

and replace y by a^x to give

$$\frac{dy}{dx} = a^x \ln a.$$

This is an example of *implicit differentiation.*

We could have obtained the same solution by initially writing a^x as a combination of a log and exp

$$\begin{aligned} y &= \exp(\ln a^x) = \exp(x \ln a) \\ y' &= \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a}\frac{d}{dx}(x \ln a) \\ &= a^x \ln a. \end{aligned}$$

Consider the earlier implicit function given by

$$4y^4 - 2y^2x^2 - yx^2 + x^2 + 3 = 0.$$

The resulting derivative will also be an implicit function. Differentiating gives

$$\begin{aligned} 16y^3y' - 2\left(2yy'x^2 + 2y^2x\right) - \left(y'x^2 + 2xy\right) &= -2x \\ \left(16y^3 - 2yx^2 - x^2\right)y' &= -2x + 4y^2x + 2xy \\ y' &= \frac{-2x + 4y^2x + 2xy}{16y^3 - 2yx^2 - x^2} \end{aligned}$$

1.4.5 Higher Derivatives

These are defined recursively;

$$\begin{aligned} f''(x) &= \frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \\ f'''(x) &= \frac{d^{\,3} f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \end{aligned}$$

and so on. For example:

$$\begin{aligned} f(x) &= 4x^3 \longrightarrow f^{\,\prime}(x) = 12x^2 \longrightarrow f''(x) = 24x \\ f'''(x) &= 24 \longrightarrow f^{(iv)}(x) = 0. \end{aligned}$$

so for any n^{th} degree polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$$

we have $f^{(n+1)}(x) = 0$.

Consider another two examples

$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \longrightarrow f''(x) = e^x \\ &\vdots \\ f^{(n)}(x) &= e^x = f(x). \end{aligned}$$

$$\begin{aligned} g(x) &= \log x \longrightarrow g'(x) = 1/x \\ g''(x) &= -1/x^2 \longrightarrow g'''(x) = 2/x^3. \end{aligned}$$

Warning

Not all functions are differentiable everywhere. For example, $1/x$ has the derivative $-1/x^2$ but only for $x \neq 0$.

Easy way is to "look for a hole", e.g. $f(x) = \dfrac{1}{x-2}$ does not exist at $x = 2$.

$x = 2$ is called a *singularity* for this function. We say $f(x)$ is *singular* at the point $x = 2$.

1.4.6 Leibniz Rule

This is the first of two rules due to Leibniz. Here it is used to obtain the n^{th} derivative of a product $y = uv$, by starting with the product rule.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \equiv uDv + vDu$$

then

$$\begin{aligned} y'' &= uD^2v + 2DuDv + vD^2u \\ y''' &= uD^3v + 3DuD^2v + 3D^2uDv + vD^3u \end{aligned}$$

and so on. This suggests (can be proved by induction)

$$D^n(uv) = uD^nv + \binom{n}{1}DuD^{n-1}v + \binom{n}{2}D^2uD^{n-2}v + \ldots + \binom{n}{r}D^ruD^{n-r}v + \ldots + vD^nu$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Example: Find the n^{th} derivative of $y = x^3 e^{ax}$.

Put $u = x^3$ and $v = e^{ax}$ and $D^n(uv) \equiv (uv)_n$, so

$$(uv)_n = uv_n + \binom{n}{1}u_1 v_{n-1} + \binom{n}{2}u_2 v_{n-2} + \binom{n}{3}u_3 v_{n-3} + \ldots\ldots.$$

$$\begin{aligned} u &= x^3;\ u_1 = 3x^2;\ u_2 = 6x;\ u_3 = 6;\ u_4 = 0 \\ v &= e^{ax};\ v_1 = ae^{ax};\ v_2 = a^2 e^{ax}; \ldots\ldots.; v_n = a^n e^{ax} \end{aligned}$$

therefore $D^n\left(x^3 e^{ax}\right) =$

$$\begin{aligned} & x^3 a^n e^{ax} + \binom{n}{1}3x^2 a^{n-1} e^{ax} + \binom{n}{2}6x a^{n-2} e^{ax} + \binom{n}{3}6a^{n-3} e^{ax} \\ = \ & e^{ax}\left(x^3 a^n + n3x^2 a^{n-1} + n(n-1)a^{n-2}3x + n(n-1)(n-2)a^{n-3}\right) \end{aligned}$$

1.4.7 Further Limits

This will be an application of differentiation. Consider the limiting case

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} \equiv \frac{0}{0} \text{ or } \frac{\infty}{\infty}$$

This is called an *indeterminate form*. Then *L' Hospitals rule* states

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow a} \frac{f'(x)}{g'(x)} = \ .... \ = \lim_{x \longrightarrow a} \frac{f^{(r)}(x)}{g^{(r)}(x)}$$

for r such that we have the indeterminate form $0/0$. If for $r+1$ we have

$$\lim_{x \longrightarrow a} \frac{f^{(r+1)}(x)}{g^{(r+1)}(x)} \longrightarrow A$$

where A is not of the form $0/0$ then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} \equiv \lim_{x \longrightarrow a} \frac{f^{(r+1)}(x)}{g^{(r+1)}(x)}.$$

Note: Very important to verify quotient has this indeterminate form before using L'Hospitals rule. Else we end up with an incorrect solution.

Examples:

1.
$$\lim_{x\to 0}\frac{\cos x+2x-1}{3x}\equiv\frac{0}{0}$$

So differentiate both numerator and denominator $\longrightarrow$

$$\lim_{x\to 0}\frac{\frac{d}{dx}(\cos x+2x-1)}{\frac{d}{dx}(3x)}=\lim_{x\to 0}\frac{-\sin x+2}{3}\neq\frac{0}{0}\to\frac{2}{3}$$

2. $\displaystyle\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos 2x}$; quotient has form $0/0$. By L' Hospital's rule we have $\displaystyle\lim_{x\to 0}\frac{e^x-e^{-x}}{2\sin 2x}$, which has indeterminate form $0/0$ again for 2nd time, so

we apply L' Hospital's rule again

$$\lim_{x \to 0} \frac{e^x + e^{-x}}{4 \cos 2x} = \frac{1}{2}.$$

3. $\lim_{x \to \infty} \frac{x^2}{\ln x} \equiv \frac{\infty}{\infty} \Rightarrow$ use L'Hospital , so $\lim_{x \to \infty} \frac{2x}{1/x} \to \infty$

4. $\lim_{x \to \infty} \frac{e^{3x}}{\ln x} \equiv \frac{\infty}{\infty} \Rightarrow \lim_{x \to \infty} 3xe^{3x} \to \infty$

5. $\lim_{x \to \infty} x^2 e^{-3x} \equiv 0.\infty$, so we convert to form ∞/∞ by writing $\lim_{x \to \infty} \frac{x^2}{e^{3x}}$,
and now use L'Hospital (differentiate twice), which gives $\lim_{x \to \infty} \frac{2}{9e^{3x}} \to 0$

6. $\lim_{x \to 0} \frac{\sin x}{x} \equiv \lim_{x \to 0} \cos x \approx 1$

What is example 6. saying?

When x is very close to 0 then $\sin x \approx x$. That is $\sin x$ can be approximated with the function x for small values.

1.5 Taylor Series

Many functions are so complicated that it is not easy to see what they look like. If we only want to know what a function looks like *locally*, we can approximate it by simpler functions: polynomials. The crudest approximation is by a constant: if $f(x)$ is continuous at x_0,

$$f(x) \approx f(x_0)$$

for x near x_0.

Before we consider this in a more formal manner we start by looking at a simple motivating example:

Consider $f(x) = e^x$.

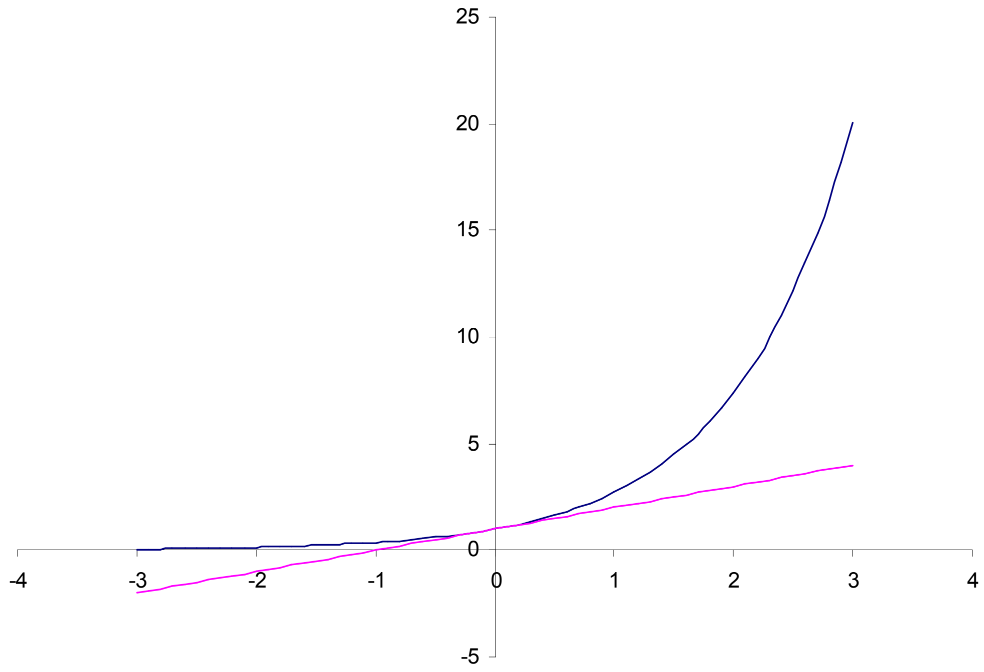
Suppose we wish to approximate this function for very small values of x (i.e. $x \longrightarrow 0$). We know at $x = 0$, $\frac{df}{dx} = 1$. So this is the gradient at $x = 0$. We can find the equation of the line that passes through a point (x_0, y_0) using

$$y - y_0 = m\,(x - x_0)\,.$$

Here $m = \frac{df}{dx} = 1$, $x_0 = 0$, $y_0 = 1$, so $y = 1 + x$, is a polynomial. What information have we ascertained from this?

If $x \longrightarrow 0$ then the point $(x, 1 + x)$ on the tangent is close to the point (x, e^x) on the graph $f\,(x)$ and hence

$$e^x \approx 1 + x$$



Suppose now that we are not that close to 0. We look for a second degree polynomial (i.e. quadratic)

$$g(x) = ax^2 + bx + c \longrightarrow g' = 2ax + b \longrightarrow g'' = 2a$$

If we want this parabola $g(x)$ to have

(i) same y intercept as f :

$$g(0) = f(0) \Longrightarrow c = 1$$

(ii) same tangent as f

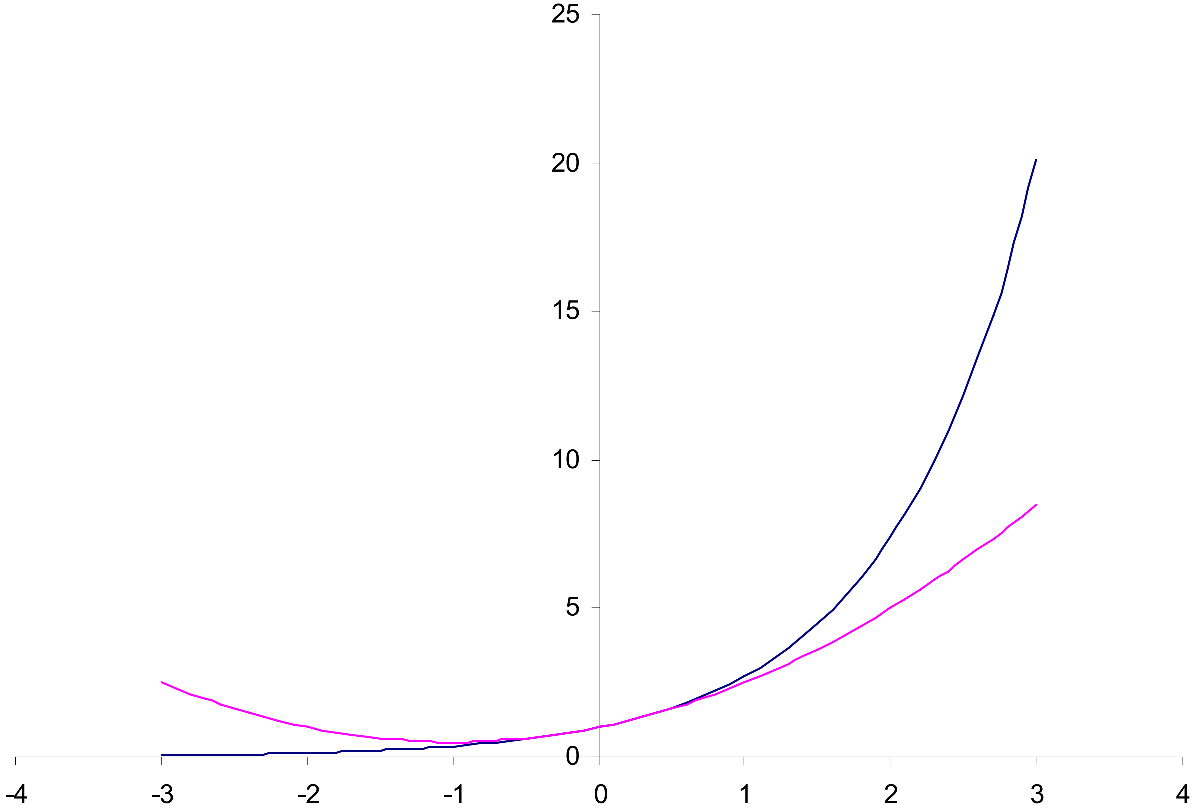
$$g'(0) = f'(0) \Longrightarrow b = 1$$

(iii) same curvature as f

$$g''(0) = f''(0) \Longrightarrow 2a = 1$$

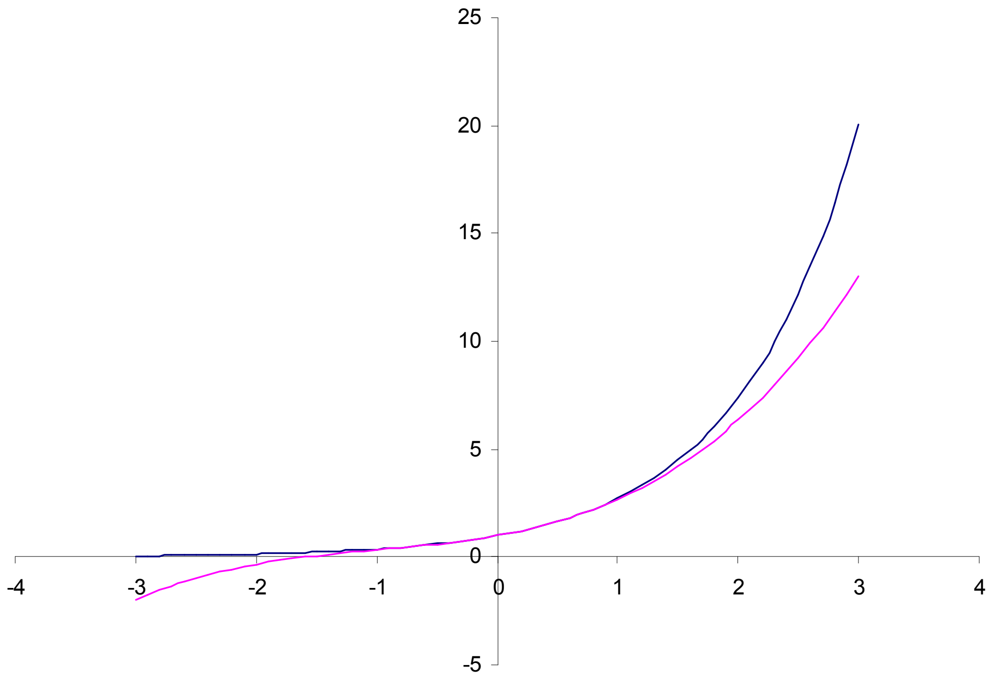
This gives

$$e^x \approx g(x) = \frac{1}{2}x^2 + x + 1$$



Moving further away we would look at a third order polynomial $h(x)$ which gives

$$e^x \approx h(x) = \frac{1}{3!}x^3 + \frac{1}{2!}x^2 + x + 1$$



and so on.

Better is to approximate by the tangent at x_0. This makes the approximation *and* its derivative agree with the function:

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0).$$

Better still is by the best fit parabola (quadratic), which makes the first two derivatives agree:

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0).$$

This process can be continued indefinitely as long as f can be differentiated often enough.

The n^{th} term is

$$\frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n,$$

where $f^{(n)}$ means the n^{th} derivative of f and $n! = n.(n-1)\ldots 2.1$ is the factorial.

$x_0 = 0$ is the special case, called *Maclaurin Series.*

Examples:

Expanding about the origin $x_0 = 0$,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}$$

Near 0, the logarithm looks like

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + (-1)^n \frac{x^{n+1}}{(n+1)!}$$

How can we obtain this? Put $f(x) = \log(1+x)$, then $f(0) = 0$

$$\begin{array}{rr} f'(x) = \frac{1}{1+x} & f'(0) = 1 \\ f''(x) = -\frac{1}{(1+x)^2} & f''(0) = -1 \\ f'''(x) = \frac{2}{(1+x)^3} & f'''(0) = 2 \\ f^{(4)}(x) = -\frac{6}{(1+x)^4} & f^{(4)}(0) = -6 \end{array}$$

Thus

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= 0 + \frac{1}{1!}x + \frac{(-1)}{2!}x^2 + \frac{1}{3!}.2x^3 + \frac{(-6)}{4!}x^4 + ... \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \end{aligned}$$

Taylor's theorem, in general, is this : If $f\ (x)$ and its first n derivatives exist (and are continuous) on some interval containing the point x_0 then

$$\begin{aligned} f(x) &= f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0) + \\ &\quad \frac{1}{2!}f''(x_0)(x-x_0)^2 + \ldots \\ &\quad + \frac{1}{(n-1)!}f^{(n-1)}(x_0)(x-x_0)^{n-1} + R_n(x) \end{aligned}$$

where $R_n(x) = (1/n!)\, f^{\ (n)}(\xi)(x-x_0)^n$, ξ is some (usually unknown) number between x_0 and x and $f^{(n)}$ is the n^{th} derivative of f.

We can expand about any point $x = a$, and shift this point to the origin, i.e. $x - x_0 \equiv 0$ and we express in powers of $(x - x_0)^n$.

So for $f(x) = \sin x$ about $x = \pi/4$ we will have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(\frac{\pi}{4}\right)}{n!} (x - \pi/4)^n$$

where $f^{(n)}\left(\frac{\pi}{4}\right)$ is the n^{th} derivative of $\sin x$ at $x_0 = \pi/4$.

As another example suppose we wish to expand $\log(1+x)$ about $x_0 = 2$, i.e. $x - 2 = 0$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(2)(x-2)^n$$

where $f^{(n)}(2)$ is the n^{th} derivative of $\log(1+x)$ evaluated at the point $x = 2$.

Note that $\log(1+x)$ does not exist for $x = -1$.

1.5.1 The Binomial Expansion

The *Binomial Theorem* is the Taylor expansion of $(1+x)^n$ where n is a positive integer. It reads:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\ .$$

We can extend this to expressions of the form

$$(1+ax)^n = 1 + n(ax) + \tfrac{n(n-1)}{2!}(ax)^2 + \tfrac{n(n-1)(n-2)}{3!}(ax)^3 + \dots\ .$$

$$(p+ax)^n = \left[p\left(1+\frac{a}{p}x\right)\right]^n = p^n\left[1 + n\left(\frac{a}{p}x\right) + \dots\dots\right]$$

The binomial coefficients are found in Pascal's triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
\quad
\begin{array}{ll}
(\mathrm{n}=0) & (1+x)^0 \\
(\mathrm{n}=1) & (1+x)^1 \\
(\mathrm{n}=2) & (1+x)^2 \\
(\mathrm{n}=3) & (1+x)^3 \\
(\mathrm{n}=4) & (1+x)^4 \\
(\mathrm{n}=5) & (1+x)^5
\end{array}
$$

and so on ...

As an example consider:

$(1+x)^3 \quad n=3 \Rightarrow 1 \quad 3 \quad 3 \quad 1 \qquad \therefore (1+x)^3 = 1+3x+3x^2+x^3$

$(1+x)^5 \quad n=5 \rightarrow (1+x)^5 = 1+5x+10x^2+10x^3+5x^4+x^5.$

If n is not an integer the theorem still holds but the coefficients are no longer integers. For example,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ .$$

and

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^2}{2!}\ldots\ .$$

$$(a+b)^k = a^k \left[1 + \tfrac{b}{a}\right]^k =$$

$$a^k \left[1 + kba^{-1} + \tfrac{k(k-1)}{2!} b^2 a^{-2} + \tfrac{k(k-1)(k-2)}{3!} b^3 a^{-3} + ..\right]$$

$$= a^k + kba^{k-1} + \tfrac{k(k-1)}{2} b^2 a^{k-2} + \tfrac{k(k-1)(k-2)}{3!} b^3 a^{k-3} + ..$$

Example: We looked at $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} \to 1$ (by L'Hospital). We can also do this using Taylor series:

$$\begin{aligned} \lim_{x \to 0} \frac{\sin x}{x} &\sim \lim_{x \to 0} \frac{x - x^3/3! + x^5/5! + ...}{x} \\ &\sim \lim_{x \to 0} \left(1 - x^2/3! + x^4/5! + ...\right) \\ &\to 1. \end{aligned}$$

1.6 Integration

1.6.1 The Indefinite Integral

The indefinite integral of $f(x)$,

$$\int f(x)\,dx,$$

is any function $F(x)$ whose derivative equals $f(x)$. Thus if

$$F(x) = \int f(x)\,dx \qquad \text{then} \qquad \frac{dF}{dx}(x) = f(x).$$

Since the derivative of a constant, C, is zero $(dC/dx = 0)$, the indefinite integral of $f(x)$ is only determined up to an arbitrary constant.

If $\dfrac{dF}{dx} = f(x)$ then

$$\frac{d}{dx}(F(x) + C) = \frac{dF}{dx}(x) + \frac{dC}{dx} = \frac{dF}{dx}(x) = f(x).$$

Thus we must always include an arbitrary constant of integration in an indefinite integral.

Simple examples are

$$\begin{aligned}
\int x^n dx &= \frac{1}{n+1}x^{n+1} + C \qquad\qquad (n \neq -1), \\
\int \frac{dx}{x} &= \log(x) + C, \quad \int e^{ax} dx = \frac{1}{a}e^{ax} + C \qquad\qquad (a \neq 0), \\
\int \cos ax dx &= \frac{1}{a}\sin ax + C, \quad \int \sin ax dx = -\frac{1}{a}\cos ax + C
\end{aligned}$$

Linearity

Integration is linear:

$$\int \left(\alpha f(x) + \beta g(x)\right) dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx$$

for constants A and B. Thus, for example

$$\begin{aligned} \int \left(Ax^2 + Bx^3\right) dx &= A \int x^2 dx + B \int x^3 dx \\ &= \frac{A}{3}x^3 + \frac{B}{4}x^4 + C, \end{aligned}$$

$$\int \left(3e^x + 2/x\right) dx = 3 \int e^x dx + 2 \int \frac{dx}{x} = 3e^x + 2\log(x) + C,$$

and so forth.

1.6.2 The Definite Integral

The **definite integral,**

$$\int_a^b f(x)\,dx,$$

is the area under the graph of $f(x)$, between $x = a$ and $x = b$, with positive values of $f(x)$ giving positive area and negative values of $f(x)$ contributing negative area. It can be computed if the indefinite integral is known. For example

$$\int_1^3 x^3 dx = \left[\frac{1}{4}x^4\right]_1^3 = \frac{1}{4}\left(3^4 - 1^4\right) = 20,$$

$$\int_{-1}^1 e^x dx = \left[e^x\right]_{-1}^1 = e - 1/e.$$

Note that the definite integral is also linear in the sense that

$$\int_a^b (Af(x) + Bg(x))\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx.$$

Note also that a definite integral

$$\int_a^b f(x)\,dx$$

does not depend on the variable of integration, x in the above, it only depends on the function f and the limits of integration (a and b in this case); the area under a curve does not depend on what we choose to call the horizontal axis.

So

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(z)\,dz.$$

We should never confuse the variable of integration with the limits of integration; a definite integral of the form

$$\int_a^x f(x)\,dx,$$

use dummy variable.

If $a < b < c$ then

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Also

$$\int_c^a f(x)\,dx = -\int_a^c f(x)\,dx.$$

1.6.3 Integration by Substitution

This involves the change of variable and used to evaluate integrals of the form

$$\int g\left(f\left(x\right)\right) f'\left(x\right) dx,$$

and can be evaluated by writing $z = f\left(x\right)$ so that $dz/dx = f'\left(x\right)$ or $dz = f'\left(x\right) dx$. Then the integral becomes

$$\int g\left(z\right) dz.$$

Examples:

$$\int \frac{x}{1+x^2} dx \quad : z = 1 + x^2 \longrightarrow dz = 2x dx$$

$$\begin{aligned} \int \frac{x}{1+x^2} dx &= \frac{1}{2} \log\left(z\right) + C = \frac{1}{2} \log\left(1 + x^2\right) + C \\ &= \log\left(\sqrt{1+x^2}\right) + C \end{aligned}$$

$$\int xe^{-x^2}dx \quad : z = -x^2 \longrightarrow dz = -2xdx$$

$$\begin{aligned}\int xe^{-x^2}dx &= -\frac{1}{2}\int e^z dz \\ &= -\frac{1}{2}e^z + C = -\frac{1}{2}e^{-x^2} + C\end{aligned}$$

$$\begin{aligned}\int \frac{1}{x}\log(x)\,dx &= \int z\ dz = \frac{1}{2}z^2 + C \\ &= \frac{1}{2}(\log(x))^2 + C\end{aligned}$$

with $z = \log(x)$ so $dz = dx/x$ and

$$\begin{aligned}\int e^{x+e^x}dx &= \int e^x e^{e^x}dx = \int e^z dz \\ &= e^z + C = e^{e^x} + C\end{aligned}$$

with $z = e^x$ so $dz = e^x dx$.

The method can be used for definite integrals too. In this case it is usually more convenient to change the limits of integration at the same time as changing the variable; this is not strictly necessary, but it can save a lot of time.

For example, consider

$$\int_1^2 e^{x^2} 2x dx.$$

Write $z = x^2$, so $dz = 2xdx$. Now consider the limits of integration; when $x = 2$, $z = x^2 = 4$ and when $x = 1$, $z = x^2 = 1$. Thus

$$\begin{aligned} \int_{x=1}^{x=2} e^{x^2} 2x dx &= \int_{z=1}^{z=4} e^z dz \\ &= [e^z]_{z=1}^{z=4} = e^4 - e^1. \end{aligned}$$

Further examples: consider

$$\int_{x=1}^{x=2} \frac{2x dx}{1+x^2}.$$

In this case we could write $z = 1 + x^2$, so $dz = 2x dx$ and $x = 1$ corresponds to $z = 2$, $x = 2$ corresponds to $z = 5$, and

$$\begin{aligned}\int_{x=1}^{x=2} \frac{2x}{1+x^2} dx &= \int_{z=2}^{z=5} \frac{dz}{z} \\ &= [\ln(z)]_{z=2}^{z=5} = \log(5) - \ln(2) \\ &= \ln(5/2)\end{aligned}$$

We can solve the same problem without change of limit, i.e.

$$\left\{\ln\left|1+x^2\right|\right\}_{x=1}^{x=2} \longrightarrow \ln 5 - \ln 2 = \ln 5/2.$$

Or consider

$$\int_{x=1}^{x=e} 2\frac{\log(x)}{x}dx$$

in which case we should choose $z = \log(x)$ so $dz = dx/x$ and $x = 1$ gives $z = 0$, $x = e$ gives $z = 1$ and so

$$\int_{x=1}^{x=e} 2\frac{\log(x)}{x}dx = \int_{z=0}^{z=1} 2z dz = \left[z^2\right]_{z=0}^{z=1} = 1.$$

When we make a substitution like $z = f(x)$ we are implicitly assuming that $dz/dx = f'(x)$ is neither infinite nor zero. It is important to remember this implicit assumption.

Consider the integral

$$\int_{-1}^{1} x^2 dx = \frac{1}{3}\left[x^3\right]_{x=-1}^{x=1} = \frac{1}{3}(1-(-1)) = \frac{2}{3}.$$

Now put $z = x^2$ so $dz = 2x dx$ or $dz = 2\sqrt{z}\, dx$ and when $x = -1$, $z = x^2 = 1$ and when $x = 1$, $z = x^2 = 1$, so

$$\int_{x=-1}^{x=1} x^2 dx = \frac{1}{2}\int_{z=1}^{z=1} \frac{dz}{\sqrt{z}} = 0$$

as the area under the curve $1/\sqrt{z}$ between $z = 1$ and $z = 1$ is obviously zero.

It is clear that $x^2 > 0$ except at $x = 0$ and therefore that

$$\int_{-1}^{1} x^2 dx = \frac{2}{3}$$

must be the correct answer. The substitution $z = x^2$ gave

$$\int_{x=-1}^{x=1} x^2 dx = \frac{1}{2} \int_{z=1}^{z=1} \frac{dz}{\sqrt{z}} = 0$$

which is obviously wrong. So why did the substitution fail?

It failed because $f\,'(x) = dz/dx = 2x$ changed signs between $x = -1$ and $x = 1$. In particular, $dz/dx = 0$ at $x = 0$, the function $z = x^2$ is not invertible for $-1 \leq x \leq 1$.

Moral: when making a substitution make sure that $dz/dx \neq 0$.

1.6.4 Integration by Parts

This is based on the product rule. In usual notation, if $y = u(x)\,v(x)$ then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

so that

$$\frac{du}{dx}v = \frac{dy}{dx} - u\frac{dv}{dx}$$

and hence integrating

$$\int \frac{du}{dx}vdx = \int \frac{dy}{dx}dx - \int u\frac{dv}{dx}dx = y(x) - \int u\frac{dv}{dx}dx + C$$

or

$$\int \frac{du}{dx}vdx = u(x)\,v(x) - \int u(x)\,\frac{dv}{dx}dx + C$$

i.e.

$$\int u'vdx = uv - \int uv'dx + C$$

This is useful, for instance, if $v(x)$ is a polynomial and $u(x)$ is an exponential.

How can we use this formula? Consider the example

$$\int xe^x dx$$

Put

$$\begin{array}{cc} v = x & u' = e^x \\ v' = 1 & u = e^x \end{array}$$

hence

$$\begin{aligned} \int xe^x dx &= uv - \int u \frac{dv}{dx} dx \\ &= xe^x - \int e^x .1 dx = e^x (x - 1) + C \end{aligned}$$

The formula we are using is the same as

$$\int v du = uv - \int u dv + C$$

Now using the same example $\int xe^x dx$

$$\begin{aligned} v = x \quad & du = e^x dx \\ dv = dx \quad & u = e^x \end{aligned}$$

and

$$\begin{aligned} \int v du &= uv - \int u dv = xe^x - \int e^x dx \\ &= e^x (x - 1) + C \end{aligned}$$

Another example

$$\int \underbrace{x^2}_{v(x)} \underbrace{e^{2x}}_{u'} dx = \underbrace{\frac{1}{2} x^2 e^{2x}}_{uv} - \int \underbrace{xe^{2x}}_{uv'} dx + C$$

and using integration by parts again

$$\int xe^{2x} dx = \frac{1}{2} xe^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{1}{4} (2x - 1) e^{2x} + D$$

so

$$\int x^2 e^{2x} dx = \frac{1}{4} \left(2x^2 - 2x + 1 \right) e^{2x} + E.$$

1.6.5 Reduction Formula

Consider the definite integral problem

$$\int_0^{\infty} e^{-t} t^n dt = I_n$$

put $v = t^n$ and $u' = e^{-t} \longrightarrow v' = nt^{n-1}$ and $u = -e^{-t}$

$$\begin{aligned} & - \left[e^{-t} t^n \right]_0^{\infty} + n \int_0^{\infty} e^{-t} t^{n-1} dt \\ &= - \left[e^{-t} t^n \right]_0^{\infty} + n I_{n-1} \\ I_n &= n I_{n-1} \\ &= n (n-1) I_{n-2} = ... = n! I_0 \end{aligned}$$

where $I_0 = \int_0^{\infty} e^{-t} dt = 1$

$$\therefore I_n = n!; \quad n \in \mathbb{Z}^+$$

I_n is called the *Gamma Function*.

1.6.6 Other Results

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

e.g.

$$\int \frac{3}{1+3x} dx = \ln |1+3x| + C$$

$$\int \frac{1}{2+7x} dx = \frac{1}{7} \int \frac{7}{2+7x} dx = \frac{1}{7} \ln |2+7x| + C$$

This allows us to state a standard result

$$\int \frac{1}{a+bx} dx = \frac{1}{b} \ln |a+bx| + C$$

How can we re-do the earlier example

$$\int \frac{x}{1+x^2} dx,$$

which was initially treated by substitution?

Partial Fractions Consider a fraction where both numerator and denominator are polynomial functions, i.e.

$$h\ (x) = \frac{f\,(x)}{g\,(x)} \equiv \frac{\sum\limits_{n=0}^{N} a_n x^n}{\sum\limits_{n=0}^{M} b_n x^n}$$

where deg $f\,(x) <$ deg $g\,(x)$, i.e. $N < M$. Then $h\ (x)$ is called a *partial fraction.* Suppose

$$\frac{c}{(x+a)\,(x+b)} \equiv \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

then writing

$$c = A\,(x+b) + B\,(x+a)$$

and solving for A and B allows us to obtain partial fractions.

The simplest way to achieve this is by setting $x = -b$ to obtain the value of B, then putting $x = -a$ yields A.

Example: $\dfrac{1}{(x-2)(x+3)}$. Now write

$$\frac{1}{(x-2)(x+3)} \equiv \frac{A}{x-2} + \frac{B}{x+3}$$

which becomes

$$1 = A(x+3) + B(x-2)$$

Setting $x = -3 \rightarrow B = -1/5; \quad x = 2 \rightarrow A = 1/5$. So

$$\frac{1}{(x-2)(x+3)} \equiv \frac{1}{5(x-2)} - \frac{1}{5(x+3)}.$$

There is another quicker and simpler method to obtain partial fractions, called the "*cover-up*" rule. As an example consider

$$\frac{x}{(x-2)\,(x+3)} \equiv \frac{A}{x-2} + \frac{B}{x+3}.$$

Firstly, look at the term $\dfrac{A}{x-2}$. The denominator vanishes for $x = 2$, so take the expression on the LHS and "cover-up" $(x-2)$. Now evaluate the remaining expression, i.e. $\dfrac{x}{(x+3)}$ for $x = 2$, which gives $2/5$. So $A = 2/5$.

Now repeat this, by noting that $\dfrac{B}{x+3}$ does not exist at $x = -3$. So cover up $(x+3)$ on the LHS and evaluate $\dfrac{x}{(x-2)}$ for $x = -3$, which gives $B = 3/5$.

Any rational expression $\dfrac{f\left(x\right)}{g\left(x\right)}$ (with degree of $f(x)$ $<$ degree of $g(x)$) such as above can be written

$$\frac{f\left(x\right)}{g\left(x\right)} \equiv F_1 + F_2 + \ \ldots\ldots.. \ + F_k$$

where each F_i has form

$$\frac{A}{(px+q)^m} \ \text{ or } \ \frac{Cx+D}{\left(ax^2+bx+c\right)^n}$$

where $\dfrac{A}{(px+q)^m}$ is written as

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \ \ldots\ldots \ + \frac{A}{(px+q)^m}$$

and $\dfrac{Cx + D}{\left(ax^2 + bx + c\right)^n}$ becomes

$$\frac{C_1 x + D_1}{ax^2 + bx + c} + \;\; ...... + \frac{C_n x + D_n}{\left(ax^2 + bx + c\right)^n}$$

Examples:

$$\frac{3x-2}{(4x-3)(2x+5)^3} \equiv \frac{A}{4x-3} + \frac{B}{2x+5} + \frac{C}{(2x+5)^2} + \frac{D}{(2x+5)^3}$$

$$\frac{4x^2+13x-9}{x(x+3)(x-1)} \equiv \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x-1)}$$

$$\frac{3x^3-18x^2+29x-4}{(x+1)(x-2)^3} \equiv \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

$$\frac{5x^2-x+2}{\left(x^2+2x+4\right)^2(x-1)} \equiv \frac{Ax+B}{x^2+2x+4} + \frac{Cx+D}{\left(x^2+2x+4\right)^2} + \frac{E}{x-1}$$

$$\frac{x^2-x-21}{\left(x^2+4\right)^2(2x-1)} \equiv \frac{Ax+B}{x^2+4} + \frac{Cx+D}{\left(x^2+4\right)^2} + \frac{E}{2x-1}$$

1.7 Complex Numbers

A complex number z is defined by $z = x + iy$ where $x,\ y \in \mathbb{R}$ and $i = \sqrt{-1}$. It follows that $i^2 = -1$.

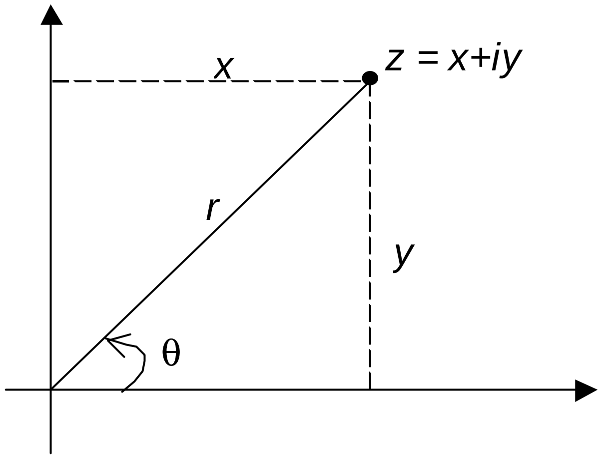
We call the $x-$ axis the real line and the $y-$ axis the imaginary line.

z may also be expressed in polar co-ordinate form as

$$z = r\,(\cos\theta + i\sin\theta)$$

where r is always positive and θ counter-clockwise from Ox.

So $x = r\cos\theta, \quad y = r\sin\theta$



modulus of z denoted $|z|$ is defined $|z| = r = +\sqrt{x^2 + y^2}$, *argument* $\theta = \text{arctan}\, \frac{y}{x}$

The set of all complex numbers is denoted $\mathbb{C}$, and for any complex number z we write $z \in \mathbb{C}$. We can think of $\mathbb{R} \subset \mathbb{C}$.

We define the *complex conjugate* of z by $\overline{z}$ where

$$\overline{z} = x - iy.$$

$\overline{z}$ is the reflection of z in the real line. So for example if $z = 1 - 2i$, then $\overline{z} = 1 + 2i$.

1.7.1 Arithmetic

Given any two complex numbers $z_1 = a + ib,\ z_2 = c + id$ the following definitions hold:

Addition & Subtraction $\quad z_1 \pm z_2 = (a \pm c) + i\,(b \pm d)$

Multiplication $\quad z_1 \times z_2 = (ac - bd) + i\,(ad + bc)$

Division $\quad \dfrac{z_1}{z_2} = \dfrac{a+ib}{c+id} = \dfrac{(ac+bd) + i\,(bc-ad)}{c^2 + d^{\,2}} = \dfrac{(ac+bd)}{c^2+d^2} + i\dfrac{(bc-ad)}{c^2+d^2}$

here we have simply multiplied by $\dfrac{c-id}{c-id}$ and note that $(c+id)\,(c-id) = c^2 + d^2$

Examples

$$z_1 = 1 + 2i, \quad z_2 = 3 - i$$

$z_1 + z_2 = (1+3) + i\,(2-1) = 4 + i$; $\quad z_1 - z_2 = (1-3) - i\,(2-(-1)) = -2 + 3i$

$$z_1 \times z_2 = (1.3 - 2. - 1) + i\,(1. - 1 + 2.3) = 5 + 5i$$

$$\frac{z_1}{z_2} = \frac{1+2i}{3-i} . \frac{3+i}{3+i} = \frac{1}{10} + i\frac{7}{10}$$

1.7.2 Complex Conjugate Identities

1. $\overline{(\overline{z})} = z$

2. $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z}_2$

3. $\overline{(z_1 z_2)} = \overline{z}_1 \overline{z}_2$

4. $z + \overline{z} \;= 2x = 2\,\mathrm{Re}\,z \qquad \Rightarrow \mathrm{Re}\,z = \dfrac{z + \overline{z}}{2}$

5. $z - \overline{z} \;= 2iy = 2i\,\mathrm{Im}\,z \qquad \Rightarrow \mathrm{Im}\,z = \dfrac{z - \overline{z}}{2i}$

6. $z.\overline{z} \;= (x + iy)\,(x - iy) = |z|^2$

7. $|\overline{z}|^2 = \overline{z}\overline{(\overline{z})} = \overline{z}z = |z|^2 \qquad \Rightarrow |\overline{z}| = |z|$

8. $\dfrac{z_1}{z_2} = \dfrac{z_1}{z_2}.\dfrac{\overline{z}_2}{\overline{z}_2} = \dfrac{z_1\overline{z}_2}{|\overline{z}_2|^2}$

9. $|z_1 z_2|^2 = |z_1|^2\,|z_2|^2$

1.7.3 Polar Form

We return to the polar form representation of complex numbers. We now introduce a new notation. If $z \in \mathbb{C}$, then

$$z = r\,(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Hence

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which is a special relationship called *Euler's Identity.* Knowing $\sin\theta$ is an odd function gives $e^{-i\theta} = \cos\theta - i\sin\theta$. Referring to the earlier polar coordinate figure, we have:

$$|z| = r, \quad \arg z = \theta$$

If

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}$$

then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \Rightarrow |z_1 z_2| = r_1 r_2 = |z_1|\,|z_2| \\ \arg(z_1 z_2) &= \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)\,. \end{aligned}$$

If $z_2 \neq 0$ then

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

and hence

$$\begin{aligned} \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \\ \arg\left(\frac{z_1}{z_2}\right) &= \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2) \end{aligned}$$

Euler's Formula Let θ be any a1ngle, then

$$\exp(i\theta) = \cos\theta + i\sin\theta.$$

We can prove this by considering the Taylor series for $\exp(x)$, $\sin x$, $\cos x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!} \tag{a}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{b}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots\ldots\ldots + (-1)^n \frac{x^{2n}}{(2n)!} \tag{c}$$

Replacing x by the purely imaginary quantity $i\theta$ in (a), we obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots\ldots\ldots\ldots + \frac{(i\theta)^n}{n!} \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots\ldots\ldots\right) + \\ &\quad i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\ldots\ldots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Note: When $\theta = \pi$ then $\exp i\pi = -1$ and $\theta = \pi/2$ gives $\exp(i\pi/2) = i$.

We can apply Euler's formula to integral problems. Consider the problem

$$\int e^x \sin x dx$$

which was simplified using the integration by parts method. We know $\operatorname{Re} e^{i\theta} = \cos\theta$, so the above becomes

$$\begin{aligned}
\int e^x \operatorname{Im} e^{ix} dx &= \int \operatorname{Im} e^{(i+1)x} dx = \operatorname{Im} \tfrac{1}{1+i} e^{(i+1)x} \\
&= e^x \operatorname{Im} \tfrac{1}{1+i} \left(e^{ix}\right) = e^x \operatorname{Im} \tfrac{1-i}{(1+i)(1-i)} \left(e^{ix}\right) \\
&= \tfrac{1}{2} e^x \operatorname{Im} (1-i) \left(e^{ix}\right) = \tfrac{1}{2} e^x \operatorname{Im} \left(e^{ix} - i e^{ix}\right) \\
&= \tfrac{1}{2} e^x \operatorname{Im} (\cos x + i \sin x - i \cos x + \sin x) \\
&= \tfrac{1}{2} e^x (\sin x - \cos x)
\end{aligned}$$

Exercise: Apply this method to solving $\int e^x \cos x dx$.

1.8 Functions of Several Variables: Multivariate Calculus

A function can depend on more than one variable. For example, the value of an option depends on the underlying asset price S (for 'spot' or 'share') and time t. We can write its value as $V(S,t)$.

The value also depends on other parameters such as the exercise price E, interest rate r and so on. Although we could write $V(S,t,E,r,...)$, it is usually clearer to leave these other variables out.

Depending on the application, the independent variables may be x and t for space and time, or two space variables x and y, or S and t for price and time, and so on.

Consider a function $z = f(x, y)$, which can be thought of as a surface in $x, \ y, \ z$ space. We can think of x and y as positions on a two dimensional grid (or as spacial variables) and z as the height of a surface above the (x, y) grid.

How do we differentiate a function $f(x, y)$ of *two* variables? What if there are more independent variables?

The **partial derivative** of $f(x, y)$ with respect to x is written

$$\frac{\partial f}{\partial x}$$

(note ∂ and not d). It is the $x-$ derivative of f *with* y *held fixed:*

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}.$$

The other partial derivative, $\partial f/\partial y$, is defined similarly but now x is held fixed:

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}.$$

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}$$

are sometimes written as f_x and f_y.

Examples

If

$$f(x, y) = x + y^2 + xe^{-y^2}$$

then

$$\frac{\partial f}{\partial x} = f_x = 1 + 0 + 1 \cdot e^{-y^2}$$

$$\frac{\partial f}{\partial y} = f_y = 0 + 2y + x \cdot (-2y)\, e^{-y^2}.$$

The convention is, treat the other variable like a constant.

Higher Derivatives

Like ordinary derivatives, these are defined recursively:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \\ \frac{\partial^2 f}{\partial y^2} &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \\ \frac{\partial^2 f}{\partial y \partial x} &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).\end{aligned}$$

If f is well-behaved, the 'mixed' partial derivatives are equal:

$$f_{xy} = f_{yx}.$$

i.e. the second order derivatives exist and are continuous.

Example:

With $f(x, y) = x + y^2 + xe^{-y^2}$ as above,

$$f_x = 1 + e^{-y^2}$$

so

$$f_{xx} = 0; \quad f_{xy} = -2ye^{-y^2}$$

Also

$$f_y = 2y - 2xye^{-y^2}$$

so

$$f_{yx} = -2ye^{-y^2}; \quad f_{yy} = 2 - 2xe^{-y^2} + 4xy^2e^{-y^2}$$

Note that $f_{xy} = f_{yx}$.

1.8.1 The Chain Rule I

Suppose that $x = x(s)$ and $y = y(s)$ and $F(s) = f\ (x(s), y(s))$. Then

$$\frac{dF}{ds}(s) = \frac{\partial f}{\partial x}(x(s), y(s))\frac{dx}{ds}(s) + \frac{\partial f}{\partial y}(x(s), y(s))\frac{dy}{ds}(s)$$

Thus if $f(x.y) = x^2 + y^2$ and $x(s) = \cos(s)$, $y(s) = \sin(s)$ we find that $F(s) = f(x(s), y(s))$ has derivative

$$\frac{dF}{ds} = -\sin(s) \cdot 2\cos(s) + \cos(s) \cdot 2\sin(s) = 0$$

which is what it should be, since $F(s) = \cos^2(s) + \sin^2(s) = 1$,

i.e. a constant.

Example: Calculate $\dfrac{dz}{dt}$ at $t = \pi/2$ where

$$z = \exp\left(xy^2\right) \qquad x = t\cos t, \quad y = t\sin t.$$

Chain rule gives

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= y^2 \exp\left(xy^2\right)(-t\sin t + \cos t) + \\ &\quad 2xy\exp\left(xy^2\right)(\sin t + t\cos t)\,. \end{aligned}$$

At $t = \pi/2$ $x = 0, \quad y = \pi/2 \Rightarrow \left.\dfrac{dz}{dt}\right|_{t=\pi/2} = -\dfrac{\pi^3}{8}.$

1.8.2 The Chain Rule II

Suppose that $x = x(u,v)$, $y = y(u,v)$ and that $F(u,v) = f\ (x(u,v),\ y(u,v))$. Then

$$\frac{\partial F}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial F}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y}.$$

This is sometimes written as

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}.$$

so is essentially a differential operator.

Example:

$$T = x^3 - xy + y^3 \quad \text{where} \quad x = r\cos\theta, \quad y = r\sin\theta$$

$$\begin{aligned}
\frac{\partial T}{\partial r} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\left(3x^2 - y\right) + \sin\theta\left(3y^2 - x\right) \\
&= \cos\theta\left(3r^2\cos^2\theta - r\sin\theta\right) + \\
&\quad \sin\theta\left(3r^2\sin^2\theta - r\cos\theta\right) \\
&= 3r^2\left(\cos^3\theta + \sin^3\theta\right) - 2r\cos\theta\sin\theta \\
&= 3r^2\left(\cos^3\theta + \sin^3\theta\right) - r\sin 2\theta.
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial T}{\partial \theta} &= \frac{\partial T}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial \theta} \\
&= -r\sin\theta\left(3x^2 - y\right) + r\cos\theta\left(3y^2 - x\right) \\
&= -r\sin\theta\left(3r^2\cos^2\theta - r\sin\theta\right) + \\
&\quad\ r\cos\theta\left(3r^2\sin^2\theta - r\cos\theta\right) \\
&= 3r^3\cos\theta\sin\theta\left(\sin\theta - \cos\theta\right) + \\
&\quad\ r^2\left(\sin^2\theta - \cos^2\theta\right). \\
&= r^2\left(\sin\theta - \cos\theta\right)\left(3r\cos\theta\sin\theta + \sin\theta + \cos\theta\right)
\end{aligned}
$$

1.8.3 Taylor for two Variables

Assuming that a function $f(x,t)$ is differentiable enough, near $x = x_0$, $t = t_0$,

$$\begin{aligned} f(x,t) &= f(x_0,t_0) + (x - x_0) f_x(x_0,t_0) + \\ &\quad (t - t_0) f_t(x_0,t_0) \\ &\quad + \frac{1}{2}\left[\begin{array}{c} (x - x_0)^2 f_{xx}(x_0,t_0) \\ +2(x - x_0)(t - t_0) f_{xt}(x_0,t_0) \\ +(t - t_0)^2 f_{tt}(x_0,t_0) \end{array}\right] + \;.... \end{aligned}$$

That is,

$$\begin{aligned} f\ (x,t) &= \text{constant} + \text{ linear} + \text{quadratic} \\ &\quad +.... \end{aligned}$$

The error in truncating this series after the second order terms tends to zero faster than the included terms. This result is particularly important for Itô's lemma in Stochastic Calculus.

Suppose a function $f = f(x, y)$ and both x, y change by a small amount, so $x \longrightarrow x + \delta x$ and $y \longrightarrow y + \delta y$, then we can examine the change in f using a two dimensional form of Taylor

$$\begin{aligned} f(x + \delta x, y + \delta y) \; = \; & f(x, y) + f_x \delta x + f_y \delta y + \\ & \frac{1}{2} f_{xx} \delta x^2 + \frac{1}{2} f_{yy} \delta y^2 + \\ & f_{xy} \delta x \delta y + O\left(\delta x^2, \delta y^2\right). \end{aligned}$$

By taking $f(x, y)$ to the lhs, writing

$$df = f(x + \delta x, y + \delta y) - f(x, y)$$

and considering only linear terms, i.e.

$$df = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$$

we obtain a formula for the *differential* or *total change* in f.